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# Level crossings of a two-parameter random walk

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## Abstract

We prove that the number  $Z(N)$  of level crossings of a two-parameter simple random walk in its first  $N \times N$  steps is almost surely  $N^{3/2+o(1)}$  as  $N \rightarrow \infty$ . The main ingredient is a strong approximation of  $Z(N)$  by the crossing local time of a Brownian sheet. Our result provides a useful algorithm for simulating the level sets of the Brownian sheet.

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## 1. Introduction

Recall that a two-parameter real-valued Brownian sheet  $W = \{W(s, t)\}_{s, t \geq 0}$  is a centered Gaussian process with covariance

$$E\{W(s, t)W(s', t')\} = \min(s, s') \times \min(t, t') \quad \forall s, t, s', t' \geq 0. \quad (1.1)$$

It is known that the level sets of  $W$  have a rich and complicated structure. For instance, if  $W^{-1}\{a\} := \{(s, t) \in \mathbf{R}_+^2 : W(s, t) = a\}$ , then it follows that with probability one,

$$\dim(W^{-1}\{a\}) = \frac{3}{2} \quad \forall a \in \mathbf{R}; \quad (1.2)$$

cf. [1, 12]. Here,  $\dim$  refers to Hausdorff dimension. Other, more delicate, features of the level sets can be found in [5–11, 13].

One expects that the level sets of two-parameter random walks are uniform in local time. Informally speaking, this and (1.2) together imply that for any reasonable discrete approximation  $\mathcal{A}_N$  of  $W^{-1}\{0\} \cap [0, 1]^2$ , one might expect that  $\#\mathcal{A}_N \approx N^{3/2}$ , as  $N \rightarrow \infty$ ; here,  $\#$  denotes cardinality, and “ $\approx$ ” stands for any reasonable notion of asymptotic equivalence.

This paper is motivated, in part, by our desire to find a good algorithm for simulating the zero-set of  $W$  inside a given box that we take to be  $[0, 1]^2$  to be concrete. A natural way to try and do this is by first performing a random-walk approximation to  $W$ , and then approximating the zero-set of  $W$  by that of the walk.

With this in mind, let  $\{X_{i,j}\}_{i,j \geq 1}$  denote an array of i.i.d. random variables with  $P\{X_{i,j} = 1\} = P\{X_{i,j} = -1\} = \frac{1}{2}$ , and consider the two-parameter random walk  $\{S_{m,n}; m, n \geq 0\}$  defined as

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n X_{i,j} \quad \forall m, n \geq 1, \quad (1.3)$$

with the added stipulation that  $S_{m,n} = 0$  whenever  $mn = 0$ . It is then possible to show that as  $N \rightarrow \infty$ ,

$$\{N^{-1}S_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}\}_{0 \leq s, t \leq 1} \Rightarrow \{W(s, t)\}_{0 \leq s, t \leq 1}, \quad (1.4)$$

where  $\Rightarrow$  denotes weak convergence in a suitable space. Here, convergence in  $\mathcal{D}_{\mathcal{D}_{\mathbf{R}}[0,1]}([0, 1])$  will do, but we will not need this fact in the sequel; see [14, Theorem 4.1.1, Chapter 6] for a variant of this statement. Suffice it to say that the factor of  $N^{-1}$  is the central limit scaling that comes from adding  $O(N^2)$  i.i.d. variates.

A natural approximation of the zero-set  $W^{-1}\{0\} \cap [0, 1]^2$  would then be the random set

$$\mathcal{Y}_N = \{(i, j) \in \{0, \dots, N\}^2 : S_{i,j} = 0\}, \quad (1.5)$$

where  $N$  is a large integer. While this algorithm is intuitively attractive, it does *not* perform well. Indeed, by the local limit theorem [19, Theorem 2.8],

$$E\{\#Y_N\} \sim (2\pi)^{-1/2} \sum_{i=1}^N \sum_{j=1}^N (ij)^{-1/2} \sim 4(2\pi)^{-1/2} N, \quad \text{as } N \rightarrow \infty. \quad (1.6)$$

One can use this, in conjunction with a monotonicity argument, to show that with probability one,

$$\lim_{N \rightarrow \infty} N^{-3/2} \#Y_N = 0. \quad (1.7)$$

In light of (1.2) and its proceeding discussion, (1.7) suggests that  $Y_N$  might be too thin to properly simulate the zero-set  $W^{-1}\{0\} \cap [0, 1]^2$  of the Brownian sheet.

In this paper, we present an alternative algorithm for simulating the zero-set of  $W$ , and show that our approximation has the correct size of  $N^{(3/2)+o(1)}$  as  $N \rightarrow \infty$ . Our suggested approximation is a natural one that is based on the “crossing numbers” of the approximating two-parameter random walk  $S$ .

A lattice point  $(i, j)$  is called a (vertical) *crossing* for the random walk  $S$  if

$$S_{ij} S_{i,j+1} < 0. \quad (1.8)$$

Define

$$\begin{aligned} \mathcal{E}_N &:= \{(i, j) \in [0, N]^2 \cap \mathbf{Z}^2 : (i, j) \text{ is a crossing}\} \\ \zeta_i(N) &:= \#\{j \in [0, N] \cap \mathbf{Z}^2 : (i, j) \text{ is a crossing}\} \\ Z(N) &:= \zeta_1(N) + \cdots + \zeta_N(N). \end{aligned} \quad (1.9)$$

In words,  $Z(N) = \#\mathcal{E}_N$  is the total number of crossings of the random walk in the first  $N \times N$  steps. We propose to show that  $\mathcal{E}_N$  is a good approximation to the zero-set of the Brownian sheet in  $[0, 1]^2$ , at least in the sense that  $Z(N) = \#\mathcal{E}_N$  is sufficiently thick in the following asymptotically sense.

**Theorem 1.1.** *With probability one,  $Z(N) = N^{(3/2)+o(1)}$  as  $N \rightarrow \infty$ .*

Fig. 1 shows the simulation of the level set of a two-parameter simple walk, together with the vertical crossings of the same random walk. The figure speaks for itself, and the Matlab code is added as a brief appendix at the end of the paper.

Throughout this paper, we write  $\log x := \ln(x \vee e)$ .

## 2. Brownian sheet and invariance

To prove Theorem 1.1, we need to analyze the crossings of the walk simultaneously at all levels. With this in mind, for each  $x \in \mathbf{R}$  we say that  $(i, j)$  is a (vertical)  $x$ -crossing for the random walk if

$$(S_{ij} - x)(S_{i,j+1} - x) < 0. \quad (2.1)$$

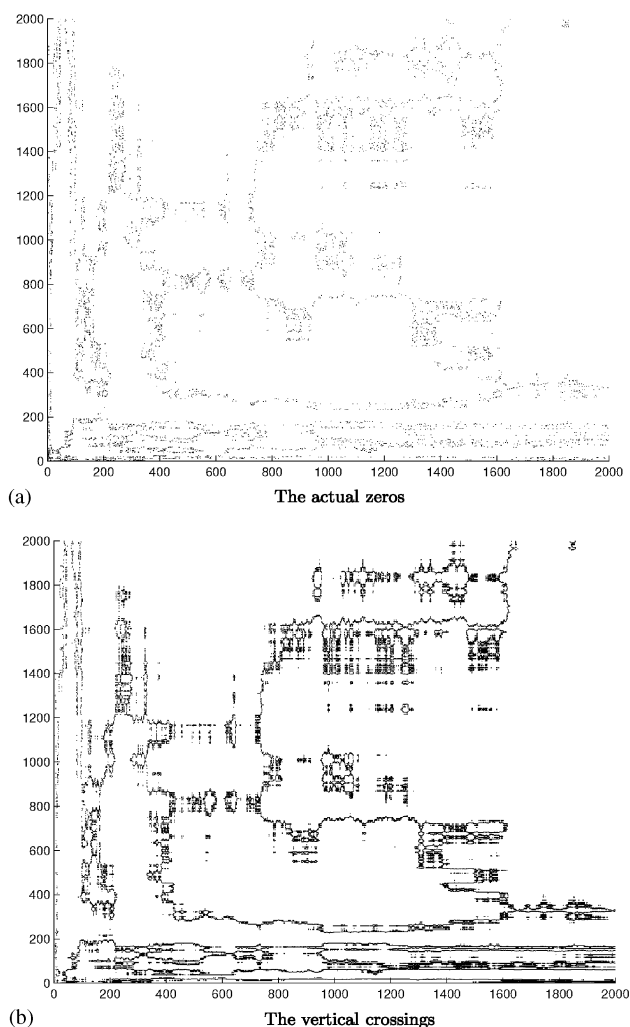


Fig. 1. The zeros vs. the vertical crossings.

Next, we can define

$$\begin{aligned}\zeta_i(x; n) &:= \#\{j : 0 \leq j < n : (i, j) \text{ is an } x\text{-crossing}\} \\ Z(x; m, n) &:= \sum_{i=0}^m \zeta_i(x; n).\end{aligned}\tag{2.2}$$

Thus,  $Z(x; m, n)$  denotes the number of  $x$ -crossings in the first  $m \times n$  steps. We remark that  $Z(N) = Z(0; N, N)$ .

When  $N$  is large, the entire process  $(x, s, t) \mapsto Z(x; \lfloor Ns \rfloor, \lfloor Nt \rfloor)$  is close to the crossing local times of a Brownian sheet that we describe next.

For any fixed  $s > 0$ ,  $\{L_s(x; t)\}_{t \geq 0}$  denotes the local time at  $x$  of the process  $t \mapsto W(s, t)$ . This is the density of the occupation measure, as described by the formula,

$$\int_{-\infty}^{+\infty} f(x) L_s(x; t) dx = \int_0^t f(W(s, u)) du. \quad (2.3)$$

Walsh [21] introduces the process  $L_s$  as the *line local time* of  $W$ .

We define the *crossing local time* of  $W$  at level  $x$  as

$$C(x; s, t) = \int_0^s \sqrt{u} L_u(x; t) du. \quad (2.4)$$

The following strong approximation constitutes a central portion of this paper.

**Theorem 2.1.** *Possibly in an enlarged probability space, there exists a coupling for the two-parameter random walk  $S$  and the Brownian sheet  $W$  such that for any  $\varepsilon > 0$  the following holds almost surely: As  $N \rightarrow \infty$ ,*

$$\begin{aligned} \max_{1 \leq m \leq N} \max_{1 \leq n \leq N} |S_{m,n} - W(m, n)| &= O(N^{1/2}(\log N)^{3/2}), \\ \max_{\substack{1 \leq m \leq N \\ 1 \leq n \leq N}} \sup_{x \in \mathbf{R}} |Z(x; m, n) - C(x; m, n)| &= o(N^{(4/3)+\varepsilon}), \end{aligned} \quad (2.5)$$

where  $Z(x; m, n)$  is the  $x$ -crossing number of the random walk, and  $C(x; s, t)$  is the crossing local time of  $W$ .

**Remark 2.2.** Theorem 2.1 implies weak convergence as a corollary. Indeed, we can note the following scaling relationship which comes only from the Brownian scaling of  $W$ :

$$\left\{ \left( \frac{W(Ns, Nt)}{N}, \frac{C(Nx; Ns, Nt)}{N^{3/2}} \right) \right\}_{s, t \geq 0, x \in \mathbf{R}} \stackrel{(d)}{=} \{(W(s, t), C(x; s, t))\}_{s, t \geq 0, x \in \mathbf{R}}, \quad (2.6)$$

where  $\stackrel{(d)}{=}$  denotes the equality of finite-dimensional distributions. Then the following is a consequence of Theorem 2.1, (2.6), and a standard estimate of modulus of continuity: As  $N \rightarrow \infty$ ,

$$\begin{aligned} &\left\{ \left( \frac{S_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}}{N}, \frac{Z(Nx; \lfloor Ns \rfloor, \lfloor Nt \rfloor)}{N^{3/2}} \right) \right\}_{s, t \in [0, 1], x \in \mathbf{R}} \\ &\Rightarrow \{(W(s, t), C(x; s, t))\}_{s, t \in [0, 1], x \in \mathbf{R}}. \end{aligned} \quad (2.7)$$

Here,  $\Rightarrow$  denotes weak convergence in the Skorohod space  $\mathcal{D}_{\mathbf{R}^2}(\mathbf{R} \times [0, 1]^2)$ .

**Remark 2.3.** We have not considered other walks than the simple ones described here. However, we suspect that, after making small adjustments, our methods would apply to other walks such as Gaussian ones. (The local time of a two-parameter Gaussian walk needs to be appropriately defined, of course.) We believe that the end-result is the same: Counting vertical crossings performs better than counting (approximate) zeros.

Let us say a few words about the proof of (2.5). Recall that  $Z(x; m, n) = \sum_{i=0}^m \zeta_i(x; n)$ , where for each  $i$ ,  $n \mapsto \zeta_i(x; n)$  is the number of  $x$ -level crossings of the one-parameter, simple random walk  $\{S_{i,j}\}_{j=0,1,2,\dots}$ . There is a rich literature for level crossings of such random walks. For example, we can appeal to [4, Theorem 1.2] to see that for each fixed  $i$ ,

$$\zeta_i(x; n) \approx \sqrt{i} L_i(x; n). \quad (2.8)$$

In words, this means that  $\zeta_i(x; n)$  can be well approximated by  $\sqrt{i} L_i(x; n)$  where  $L_i(x; n)$  is the local time of a Brownian motion with infinitesimal variance  $i$ . One might then conjecture that such local times can be embedded in the line local times  $L_i$  of a single Brownian sheet  $W$ ; cf. (2.3). If this were so, then

$$\begin{aligned} Z(x; m, n) &= \sum_{i=0}^m \zeta_i(x; n) \approx \sum_{i=0}^m \sqrt{i} L_i(x; n) \\ &\approx \int_0^m \sqrt{s} L_s(x; n) ds = C(x; m, n). \end{aligned} \quad (2.9)$$

Such a route is fraught with technical difficulties. For one, it is not clear why (2.8) should hold jointly for all  $i$  if  $L_i$  is the line local time of a single Brownian sheet. Moreover, the rate of approximation in (2.8) depends in a subtle way on  $i$ , and it is not at all clear what information can be gleaned from this about the rate of approximation in (2.9).

Our method provides an approach that is based on our attempt at solving the following loosely stated problem: “How close are the local times  $L(X_1)$  and  $L(X_2)$  of two processes  $X_1$  and  $X_2$  if  $X_1 \approx X_2$ , and if  $L(X_1)$  and  $L(X_2)$  are sufficiently smooth?” Interestingly enough, our solution to the mentioned problem uses (2.8), but only for a fixed value of  $i$ .

The remainder of the paper is organized as follows: The two parts of Theorem 2.1, namely the two claims in (2.5), are proved in distinct sections. The first assertion of (2.5) is straightforward; it is proved in Section 3. We prove the second assertion of (2.5) in Section 4. We do this by means of two technical lemmas. The said lemmas themselves are proved, respectively, in Sections 5 and 6. Finally, we derive Theorem 1.1 in Section 7. Our derivation involves an application of (2.5) and a recent estimate of the authors on the explosion rate of the local time along lines of  $W$  [15].

### 3. Proof of Theorem 2.1 (2.5): first part

Our proof of the first part of (2.5) relies on *Bernstein’s inequality* that we now recall; cf. [20, p. 855].

Let  $\{\eta_k\}_{k \geq 1}$  be a sequence of independent mean-zero variables such that for some  $c > 0$ ,  $E\{|\eta_k|^n\} \leq \frac{1}{2} v_k n! c^{n-2}$  for all  $n \geq 2$ . Then, for any  $x > 0$  and  $n \geq 1$ ,

$$P\left\{\left|\sum_{k=1}^n \eta_k\right| \geq x\right\} \leq 2 \exp\left(-\frac{1}{2} \frac{x^2}{\sum_{k=1}^n v_k + cx}\right). \quad (3.1)$$

Fix  $j \geq 1$ , and consider the sequence  $\{X_{ij}\}_{i,j \geq 1}$  of i.i.d. random variables. According to [16], possibly in an enlarged probability space, there exists a standard Brownian motions  $W_j$  such that for all  $x > 0$  and  $m \geq 1$ ,

$$P\left\{\left|\sum_{i=1}^m X_{ij} - W_j(m)\right| > c_1 \log m + x\right\} \leq c_2 e^{-c_3 x}, \quad (3.2)$$

where  $c_1, c_2$  and  $c_3$  are constants that do not depend on  $x, m$ . Since  $\{X_{ij}\}_{i,j \geq 1}$  are i.i.d., we can arrange things so that  $\{W_j\}_{j \geq 0}$  are independent processes.

Now, let us fix  $m \geq 1$ , and consider the process  $\eta = \{\eta_j\}_{j \geq 1}$  where  $\eta_j = \sum_{i=1}^m X_{ij} - W_j(m)$ . Clearly,  $\eta$  is a sequence of independent (but not necessarily i.i.d.) mean-zero variables. According to [16] for any  $n \geq 2$  and  $j \geq 1$ ,

$$\begin{aligned} E\{|\eta_j|^n\} &= \int_0^\infty n y^{n-1} P\{|\eta_j| > y\} dy \\ &\leq \int_0^{c_1 \log m} n y^{n-1} dy + \int_0^\infty n (c_1 \log m + x)^{n-1} c_2 e^{-c_3 x} dx. \end{aligned} \quad (3.3)$$

The first integral equals  $(c_1 \log m)^n$ , whereas the second is

$$\begin{aligned} &\left(\int_0^{c_1 \log m} + \int_{c_1 \log m}^\infty\right) n (c_1 \log m + x)^{n-1} c_2 e^{-c_3 x} dx \\ &\leq n (2c_1 \log m)^{n-1} c_2 \int_0^{c_1 \log m} e^{-c_3 x} dx + n c_2 \int_{c_1 \log m}^\infty (2x)^{n-1} e^{-c_3 x} dx \\ &\leq n (2c_1 \log m)^{n-1} c_2 \int_0^\infty e^{-c_3 x} dx + n c_2 \int_0^\infty (2x)^{n-1} e^{-c_3 x} dx \\ &\leq \frac{n (2c_1 \log m)^{n-1} c_2}{c_3} + \frac{n! c_2 2^{n-1}}{(c_3)^n}. \end{aligned} \quad (3.4)$$

Therefore, there exists an absolute constant  $c_4$  such that

$$E(|\eta_j|^n) \leq \frac{c_4 (\log m)^2}{2} (c_4 \log m)^{n-2} n! \quad \forall n \geq 2. \quad (3.5)$$

By Bernstein's inequality (cf. (3.1)), for any  $x > 0$  and  $m, n \geq 1$ ,

$$P\left\{\left|\sum_{j=1}^n \sum_{i=1}^m X_{ij} - \sum_{j=1}^n W_j(m)\right| \geq x\right\} \leq 2 \exp\left(-\frac{1}{2c_4} \frac{x^2}{n(\log m)^2 + x \log m}\right). \quad (3.6)$$

It is manifest that  $\{W(j+1, k) - W(j, k)\}_{j,k \geq 0}$  has the same law as  $\{W_j(k)\}_{j,k \geq 0}$ . Standard methods can then be used to prove that we can embed  $\{\sum_{j=1}^\ell W_j(k)\}_{j,k \geq 1}$  in a two-parameter Brownian sheet  $W$ , although we may possibly need to work in an enlarged probability space. (In the paper, we often use several embedding schemes in enlarged spaces. This can be justified by a coupling argument as in [3, p. 53].)

Therefore, for any  $x > 0$  and  $N \geq 1$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{1 \leq m \leq N} \max_{1 \leq n \leq N} |S_{m,n} - W(m,n)| > x \right\} \\ & \leq 2N^2 \exp \left( -\frac{1}{2c_4} \frac{x^2}{N(\log N)^2 + x \log N} \right). \end{aligned} \quad (3.7)$$

Take  $x = (7c_4)^{1/2} N^{1/2} (\log N)^{3/2}$ , so that the expression on the right-hand side is summable in  $N$ . Applying the Borel–Cantelli lemma readily yields the first assertion of (2.5).  $\square$

#### 4. Proof of Theorem 2.1 (2.5): second part

In this section, we prove the second assertion of (2.5) of Theorem 2.1. This is done by virtue of two technical estimates—Propositions 4.1 and 4.2—as well as two supporting lemmas.

Our first proposition controls the oscillations of the process  $C$ , viz.,

**Proposition 4.1.** *For any  $\alpha, \varepsilon > 0$ , the following holds almost surely: As  $N \rightarrow \infty$ ,*

$$\max_{1 \leq m \leq N} \max_{1 \leq n \leq N} \sup_{|x-y| \leq N^\alpha} |C(x; m, n) - C(y; m, n)| = o(N^{1+\max(\alpha/2, 1/4)+\varepsilon}). \quad (4.1)$$

Theorem 2.1 asserts that the processes  $Z$  and  $C$  are asymptotically close to one another. The following analogue of Proposition 4.1 states that their asymptotic moduli of continuity are also close, viewed on an appropriate scale.

**Proposition 4.2.** *Given  $\alpha, \varepsilon > 0$ , the following holds almost surely: As  $N \rightarrow \infty$ ,*

$$\max_{0 \leq m \leq N} \max_{1 \leq n \leq N} \sup_{|x-y| \leq N^\alpha} |Z(x; m, n) - Z(y; m, n)| = o(N^{1+\max(\alpha/2, 1/4)+\varepsilon}). \quad (4.2)$$

We postpone proving these Propositions until Sections 5 and 6, respectively. In the remainder of this section, we use the latter propositions to prove the second assertion of (2.5) of Theorem 2.1.

**Lemma 4.3.** *For any fixed  $\theta > \frac{1}{2}$ , the following is almost surely valid:*

$$\max_{0 \leq i, j \leq N} \sup_{i \leq u \leq i+1} \sup_{j \leq v \leq j+1} |W(u, v) - W(i, j)| \leq N^\theta \quad \forall N \text{ large}. \quad (4.3)$$

**Proof.** We can appeal to the proof of [18, Lemma 1.2] to deduce that for any  $a, b, \lambda > 0$ ,

$$\mathbb{P} \left\{ \sup_{(s,t) \in [0,a] \times [0,b]} |W(s, t)| > \lambda \right\} \leq 4\mathbb{P}\{|W(a, b)| > \lambda\}. \quad (4.4)$$



Therefore, for any  $i, j \leq N$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{i \leq u \leq i+1} \sup_{j \leq v \leq j+1} |W(u, v) - W(i, j)| > N^\theta \right\} \\ & \leq 4\mathbb{P}\{|W(1, 1)| > N^\theta\} + 4\mathbb{P}\{|W(i, 1)| > N^\theta\} + 4\mathbb{P}\{|W(1, j)| > N^\theta\} \\ & \leq 12 \exp\left(-\frac{1}{2} N^{2\theta-1}\right). \end{aligned} \quad (4.5)$$

Consequently,

$$\begin{aligned} & \sum_{N \geq 1} \mathbb{P} \left\{ \max_{0 \leq i, j \leq N} \sup_{i \leq u \leq i+1} \sup_{j \leq v \leq j+1} |W(u, v) - W(i, j)| > N^\theta \right\} \\ & \leq 12 \sum_{N \geq 1} N^2 \exp\left(-\frac{1}{2} N^{2\theta-1}\right), \end{aligned} \quad (4.6)$$

which is finite. The Borel–Cantelli lemma completes our verification.  $\square$

Our final lemma is an almost-sure uniform bound for  $Z$ .

**Lemma 4.4.** *For any fixed  $\varepsilon > 0$ , the following holds almost surely: As  $N \rightarrow \infty$ ,*

$$\sup_{x \in \mathbf{R}} Z(x; N, N) = o(N^{(3/2)+\varepsilon}). \quad (4.7)$$

**Proof.** Applying the local limit theorem, it is not hard to see that

$$\mathbb{E}\{Z(0; N, N)\} = \sum_{i=0}^N \sum_{j=0}^{N-1} \mathbb{P}\{S_{i,j} S_{i,j+1} < 0\} \leq \sum_{i=0}^N \sum_{j=0}^{N-1} \frac{c_5}{\sqrt{j+1}} \leq c_6 N^{3/2}, \quad (4.8)$$

where  $c_5$  and  $c_6$  are two unimportant constants; cf. [19, Theorem 2.8] for the local limit theorem. To this, we apply Markov’s inequality to obtain

$$\mathbb{P}\{Z(0; N, N) > N^{(3/2)+\varepsilon}\} \leq c_6 N^{-\varepsilon}. \quad (4.9)$$

Let  $N_k = \lfloor k^{2/\varepsilon} \rfloor$  and use the Borel–Cantelli lemma to see that, with probability one, for all large  $k$ ,  $Z(0; N_k, N_k) \leq N_k^{(3/2)+\varepsilon}$ . By the monotonicity of  $N \mapsto Z(0; N, N)$ ,

$$Z(0; N, N) = O(N^{(3/2)+\varepsilon}), \quad \text{a.s.} \quad (4.10)$$

Proposition 4.2 then shows that

$$\sup_{|x| \leq N^{1+\varepsilon}} |Z(x; N, N)| = o(N^{(3/2)+2\varepsilon}), \quad \text{a.s.} \quad (4.11)$$

It remains to replace  $\sup_{|x| \leq N^{1+\varepsilon}}$  by  $\sup_{x \in \mathbf{R}}$  in the above.

By the law of the iterated logarithm law for Brownian sheet,

$$\limsup_{s, t \rightarrow \infty} \frac{|W(s, t)|}{\sqrt{4st \log \log(st)}} = 1, \quad \text{a.s.}; \quad (4.12)$$

cf. [22]. This, together with (2.5), implies that

$$\max_{1 \leq m \leq N} \max_{1 \leq n \leq N} |S_{m,n}| = o(N^{1+\varepsilon}), \quad \text{a.s.} \quad (4.13)$$

Therefore, with probability one, when  $N$  is sufficiently large,  $Z(x; N, N) = 0$  for all  $|x| \geq N^{1+\varepsilon}$ . Accordingly, (4.11) implies our lemma since  $\varepsilon > 0$  is arbitrary.  $\square$

We are in position for presenting our

**Proof of Theorem 2.1 (2.5): second part.** By our definition (cf. 2.4) of the crossing local time  $C(x; s, t)$ , for any Borel set  $A \subseteq \mathbf{R}$ ,

$$\int_A C(a; s, t) da = \int_0^s du \int_0^t dv \sqrt{u} \mathbf{1}_{\{W(u,v) \in A\}}. \quad (4.14)$$

In particular, for any  $x \in \mathbf{R}$  and  $\beta > \frac{1}{2}$ ,

$$\int_x^{x+N^\beta} C(a; m, n) da = \int_0^m du \int_0^n dv \sqrt{u} \mathbf{1}_{\{x \leq W(u,v) \leq x+N^\beta\}}. \quad (4.15)$$

We apply Proposition 4.1 to this and deduce that with probability one the following exists for all  $N$  large: For all  $m, n \leq N$ ,

$$\begin{aligned} N^\beta C(x; m, n) &+ o(N^{\beta+1+\max(\beta/2, 1/4)+\varepsilon}) \\ &= \int_0^m du \int_0^n dv \sqrt{u} \mathbf{1}_{\{x \leq W(u,v) \leq x+N^\beta\}}. \end{aligned} \quad (4.16)$$

Lemma 4.3 then shows us that almost surely, as  $N \rightarrow \infty$ ,

$$\begin{aligned} &\sum_{i=0}^m \sum_{j=0}^n \sqrt{i} \mathbf{1}_{\{x+N^\theta \leq W(i,j) \leq x+N^\beta-N^\theta\}} + O(N^{3/2}) \\ &\leq \int_0^m du \int_0^n dv \sqrt{u} \mathbf{1}_{\{x \leq W(u,v) \leq x+N^\beta\}} \\ &\leq \sum_{i=0}^m \sum_{j=0}^n \sqrt{i} \mathbf{1}_{\{x-N^\theta \leq W(i,j) \leq x+N^\beta+N^\theta\}} + O(N^{3/2}). \end{aligned} \quad (4.17)$$

On the other hand, according to (2.5), for any  $a < b$ , almost surely as  $N \rightarrow \infty$ ,

$$\mathbf{1}_{\{a+N^\theta \leq S_{ij} \leq b-N^\theta\}} \leq \mathbf{1}_{\{a \leq W(i,j) \leq b\}} \leq \mathbf{1}_{\{a-N^\theta \leq S_{ij} \leq b+N^\theta\}}, \quad (4.18)$$

uniformly in  $i, j \leq N$ . Recall  $Z(x; m, n)$  from (2.2) and note that

$$\int_A Z(a; m, n) da = \sum_{i=0}^m \sum_{j=0}^n \sqrt{i} \mathbf{1}_{\{S_{ij} \in A\}}. \quad (4.19)$$

From this it follows that with probability one, as  $N \rightarrow \infty$ ,

$$\begin{aligned} \int_{x-2N^\theta}^{x-2N^\theta+N^\beta} Z(a; m, n) \, da &\leq N^\beta C(x; m, n) + o(N^{\beta+1+\max(\beta/2, 1/4)+\varepsilon}) \\ &\leq \int_{x-2N^\theta}^{x+2N^\theta+N^\beta} Z(a; m, n) \, da, \end{aligned} \quad (4.20)$$

uniformly in  $m, n \leq N$ . Combining Proposition 4.2 with the above, we arrive at

$$\begin{aligned} (N^\beta - 4N^\theta)Z(x; m, n) &\leq N^\beta C(x; m, n) + o(N^{\beta+1+\max(\beta/2, 1/4)+\varepsilon}) \\ &\leq (N^\beta + 4N^\theta)Z(x; m, n), \end{aligned} \quad (4.21)$$

uniformly in  $m, n \leq N$  and in  $x \in \mathbf{R}$ . Consequently, by Lemma 4.4, for any  $\varepsilon > 0$ ,

$$\begin{aligned} \max_{0 \leq m \leq N} \max_{1 \leq n \leq N} \sup_{x \in \mathbf{R}} |Z(x; m, n) - C(x; m, n)| \\ \leq 4N^{-(\beta-\theta)} \sup_{x \in \mathbf{R}} Z(x; N, N) + o(N^{1+\max(\beta/2, 1/4)+\varepsilon}) \\ = o(N^{-(\beta-\theta-(3/2)-\varepsilon)}) + o(N^{1+\max(\beta/2, 1/4)+\varepsilon}). \end{aligned} \quad (4.22)$$

We could take  $\beta = \frac{2}{3} + \varepsilon$  and  $\theta = \frac{1}{2} + \varepsilon$ . Since  $\varepsilon > 0$  can be chosen arbitrarily small, this yields (2.5).  $\square$

## 5. Proof of Proposition 4.1

Let us start by fixing the basic notation in this section. For any real-valued random variable  $\eta$ , we write

$$\|\eta\|_p = \{E[|\eta|^p]\}^{1/p}, \quad \text{and} \quad \langle \eta \rangle_{\text{Orl}} = \inf\{a > 0 : E[\exp(a^{-1}|\eta|)] \leq 2\}. \quad (5.1)$$

They stand for the  $L^p(\mathbf{P})$ -norm and the Orlicz (pseudo-)norm (associated with the convex function  $f(x) = e^x - 1$ ) of  $\eta$ , respectively. (To be precise,  $\langle \cdot \rangle_{\text{Orl}}$  is *not* a norm, but it is equivalent to one.) The two norms are related to each other as follows: There exists an absolute constant  $c_7$  such that

$$\langle \eta \rangle_{\text{Orl}} \leq c_7 \sup_{m \geq 1} \frac{\|\eta\|_m}{m}. \quad (5.2)$$

Our proof of Proposition 4.1 is based on the following convenient formulation of Dudley's metric entropy theorem.

**Lemma 5.1** (Lacey [17, Theorem 3.1]). *Let  $\{X(t); t \in T\}$  be a real-valued stochastic process, and let  $d_X(s, t) = \langle X_s - X_t \rangle_{\text{Orl}}$  be the natural pseudo-metric on  $T$  induced by  $X$ . Let  $\mathcal{N}(r) = N(r, T, d_X)$  be the minimum number of  $d_X$ -balls of radius  $r$  needed to cover  $T$ , and let*

$$D := \sup_{(s,t) \in T^2} d_X(s, t) \quad \text{and} \quad m_D := \int_0^D \log \mathcal{N}(r) \, dr. \quad (5.3)$$

If  $m_D < \infty$ , then there exists a universal constant  $c_8$ , such that for all  $\lambda > 0$ ,

$$\mathbb{P} \left\{ \sup_{(s,t) \in T^2} |X_s - X_t| > c_8 (\lambda + D + m_D) \right\} \leq c_8 e^{-\lambda/D}. \quad (5.4)$$

Fix  $1 \leq i, n \leq N$ , and choose  $k \in \{0, \pm 1, \pm 2, \dots, \pm k_{\max}\}$  where  $k_{\max} := \lfloor N^2/n^{1/2} \rfloor$ . We intend to apply Fact 5.1 to the process  $\{L_s(x; n); (s, x) \in T = [i, i+1] \times [k\sqrt{n}, (k+1)\sqrt{n}]\}$ , where  $L_s(x; n)$  is the line local time of the Brownian sheet  $W$ . This was defined in (2.3).

We begin by estimating the entropy number  $\mathcal{N}(r)$  induced by  $d_X$ . This will be done in successive steps.

Choose some  $(s, x) \in T$  and  $(t, y) \in T$ . In order to bound  $d_X((s, x), (t, y))$ , we start by estimating  $\|L_s(x; n) - L_t(y; n)\|_p$  for  $p \geq 1$ . This can be reduced to estimating

$$I_1(p) = \|L_s(x; n) - L_s(y; n)\|_p, \quad \text{and} \quad I_2(p) = \|L_s(y; n) - L_t(y; n)\|_p. \quad (5.5)$$

**Lemma 5.2.** *For every  $v \in (0, \frac{1}{2})$ , there exists a constant  $c_9 = c_9(v) \in (0, \infty)$  such that for all  $x, y \in \mathbf{R}$ , for all integers  $i, n \geq 1$ ,  $s \in [i, i+1]$ , and  $p \geq 1$ ,*

$$I_1(p) \leq c_9 p \frac{n^{1/2(1-v)}}{i^{1/2(1+v)}} |x - y|^v. \quad (5.6)$$

**Proof.** Writing  $\stackrel{(d)}{=}$  for equality of distributions, and use scaling to deduce that

$$L_s(x; n) - L_s(y; n) \stackrel{(d)}{=} \sqrt{\frac{n}{s}} \left[ L_1\left(\frac{x}{\sqrt{sn}}, 1\right) - L_1\left(\frac{y}{\sqrt{sn}}, 1\right) \right]. \quad (5.7)$$

Since  $L_1(x, t)$  is a standard Brownian local time, we can use the following inequality of [2]: For any  $v \in [0, \frac{1}{2})$  and  $p \geq 1$ ,

$$c_9(v) = \sup_{p \geq 1} \frac{1}{p} \left\| \sup_{0 \leq t \leq 1} \sup_{x \neq y} \frac{|L_1(x, t) - L_1(y, t)|}{|x - y|^v} \right\|_p < \infty. \quad (5.8)$$

Consequently, and writing  $c_9 = c_9(v)$  for brevity, we have

$$I_1(p) \leq c_9 p \frac{n^{1/2(1-v)}}{s^{1/2(1+v)}} |x - y|^v. \quad (5.9)$$

Since  $s \in [i, i+1]$ , this has the desired result.  $\square$

Our estimate for  $I_2(p)$  of (5.5) is derived by similar methods.

**Lemma 5.3.** *For every  $\lambda \in (0, \frac{1}{4})$ , there exists a constant  $c_{10} = c_{10}(\lambda) \in (0, \infty)$  such that for all  $p \geq 1$ , integers  $n, i \geq 1$ , and  $s, t \in [i, i+1]$ ,*

$$I_2(p) \leq c_{10} p n^{1/2} i^{-\lambda - (1/2)} (t - s)^\lambda. \quad (5.10)$$

**Proof.** We recall the following result of [17, Proposition 4.2]: For any  $p \geq 1$ ,  $0 < h < 1$  and  $0 < \lambda < \frac{1}{4}$ ,

$$\sup_{x \in \mathbf{R}} \|L_{1+h}(x; 1) - L_1(x; 1)\|_p \leq c_{10} h^\lambda p. \quad (5.11)$$

Assuming  $s \leq t$  without loss of generality, and using the scaling property, we have: For all  $i \leq s \leq t \leq i + 1$  and  $y \in \mathbf{R}$ ,

$$I_2(p) = \sqrt{\frac{n}{s}} \left\| L_1\left(\frac{y}{\sqrt{sn}}, 1\right) - L_{t/s}\left(\frac{y}{\sqrt{sn}}, 1\right) \right\|_p \leq c_{10} p n^{1/2} s^{-\lambda - (1/2)} (t - s)^\lambda. \quad (5.12)$$

Apply this to  $s, t \in [i, i + 1]$  to finish.  $\square$

**Lemma 5.4.** For every  $v \in (0, \frac{1}{2})$ , there exists a constant  $c_{12} = c_{12}(v) \in (0, \infty)$  such that for all integers  $i, n \geq 1$ ,  $x, y \in \mathbf{R}$ , and  $(s, t) \in [i, i + 1]^2$ ,

$$\langle L_s(x; n) - L_t(y; n) \rangle_{\text{On}} \leq c_{12} \frac{n^{1/2(1-v)} |x - y|^v + n^{1/2} |t - s|^{1/2v}}{i^{1/2(1+v)}}. \quad (5.13)$$

**Proof.** We choose  $\lambda = \frac{v}{2}$ , and appeal to Lemmas 5.2 and 5.3, as well as Minkowski's inequality to deduce that

$$\begin{aligned} \|L_s(x; n) - L_t(y; n)\|_p &\leq I_1(p) + I_2(p) \\ &\leq c_{11} p \frac{n^{1/2(1-v)} |x - y|^v + n^{1/2} (t - s)^{1/2v}}{i^{1/2(1+v)}}, \end{aligned} \quad (5.14)$$

where  $c_{11} := c_9 + c_{10}$ . This immediately yields our lemma.  $\square$

We now use this to estimate the asymptotic modulus of continuity of the line local times.

**Lemma 5.5.** Given a fixed  $v \in (0, \frac{1}{2})$ , there exists a constant  $c_{16} = c_{16}(v) \in (0, \infty)$  such that almost surely for all large  $N$ , and  $1 \leq i, n \leq N$ ,

$$\sup_{(s,t) \in [i, i+1]^2} \sup_{x \in \mathbf{R}} |L_s(x; n) - L_t(x; n)| \leq c_{16} \frac{N^{1/2} \log N}{i^{1/2(1+v)}}. \quad (5.15)$$

**Proof.** We prove this by applying Lemma 5.1 to the process  $(x, s) \mapsto X_{x,s} = L_s(x; n)$ , where  $n \geq 1$  is an arbitrary integer.

Recall  $D$  and  $m_D$  from (5.3), and note that thanks to Lemma 5.4,

$$D \leq c_{12} \frac{n^{1/2(1-v)} (\sqrt{n})^v + n^{1/2}}{i^{1/2(1+v)}} \leq 2c_{12} \frac{N^{1/2}}{i^{1/2(1+v)}}. \quad (5.16)$$

Next, we estimate  $m_D$ : Owing to Lemma 5.4, the minimum number  $\mathcal{N}(r)$  of  $d_X$ -balls of radius  $r$  needed to cover  $T$  satisfies

$$\mathcal{N}(r) \leq c_{13} n^{1/2} \left( \frac{1/2n^{(3-v)}}{r^3 i^{3/2(1+v)}} \right)^{1/v} \leq c_{13} N^{1/2} \left( \frac{N^{1/2(3-v)}}{r^3} \right)^{1/v}. \quad (5.17)$$

Thus,  $m_D \leq \int_0^D \log(\mathcal{N}(r)) dr \leq c_{14} D \log N$ , where  $c_{14}$  depends only on  $v$ . Applying Lemma 5.1 with  $\lambda := 6D \log N$  yields the following:

$$\begin{aligned} & \mathbf{P}\{A_{ikn}\} \leq c_8 N^{-6}, \quad \text{where} \\ & A_{ikn} := \left\{ \sup_{(s,t) \in [i,i+1]^2} \sup_{(x,y) \in [k\sqrt{n},(k+1)\sqrt{n}]^2} |L_s(x;n) - L_t(y;n)| > c_{15} D \log N \right\}, \end{aligned} \quad (5.18)$$

and  $c_{15} := c_8(7 + c_{14})$ . Therefore,

$$\mathbf{P}\left\{ \bigcup_{i=1}^N \bigcup_{n=1}^N \bigcup_{|k| \leq k_{\max}} A_{ikn} \right\} \leq N^2(2N^2 + 1) \frac{c_8}{N^6} \leq \frac{3c_8}{N^2}. \quad (5.19)$$

By the Borel–Cantelli lemma, almost surely for all large  $N$ ,  $1 \leq i, n \leq N$ ,  $(s, t) \in [i, i+1]^2$ , and  $|x| \leq N^2$ , we have

$$|L_s(x;n) - L_t(x;n)| \leq c_{15} D \log N \leq c_{16} \frac{N^{1/2} \log N}{i^{1/2(1+v)}}, \quad (5.20)$$

with  $c_{16} = 2c_{12}c_{15}$ . On the other hand, it follows from (4.12) that with probability one, when  $N$  is sufficiently large,  $\sup_{0 \leq s \leq N} \sup_{0 \leq t \leq N} |W(s, t)| < N^2$ , so that  $L_s(x;n) = 0$  for  $s, n \leq N$  and  $|x| > N^2$ . Our lemma follows as a consequence.  $\square$

We present one final supporting lemma.

**Lemma 5.6.** *If  $\alpha, \varepsilon > 0$  and  $v \in (0, \frac{1}{2})$  are held fixed, then with probability one for all large  $N$  and all  $i, n \leq N$ ,*

$$\sup_{|x-y| \leq N^\alpha} |L_i(x;n) - L_i(y;n)| \leq i^{-1/2(1+v)} N^{\alpha v + 1/2(1-v) + \varepsilon}. \quad (5.21)$$

**Proof.** Let  $\alpha > 0$  and use scaling to see that for each  $i$  and  $n$ ,  $\sup_{|x-y| \leq N^\alpha} |L_i(x;n) - L_i(y;n)|$  is distributed as  $(n/i)^{1/2} \sup_{|x-y| \leq N^\alpha/\sqrt{in}} |L_1(x;1) - L_1(y;1)|$ . Hence, for any  $b > 0$  and  $p \geq 1$ ,

$$\begin{aligned} & \mathbf{P}\left\{ \sup_{|x-y| \leq N^\alpha} |L_i(x;n) - L_i(y;n)| > b \right\} \\ & \leq b^{-p} \left( \frac{n}{i} \right)^{p/2} \left\| \sup_{|x-y| \leq N^\alpha/\sqrt{in}} |L_1(x;1) - L_1(y;1)| \right\|_p^p \end{aligned}$$

$$\begin{aligned} &\leq b^{-p} \left(\frac{n}{i}\right)^{p/2} \left(\frac{N^\alpha}{\sqrt{in}}\right)^{vp} \left\| \sup_{x \neq y} \frac{|L_1(x; 1) - L_1(y; 1)|}{|x - y|^v} \right\|_p^p \\ &\leq (pc_9(v))^p b^{-p} \left(\frac{n}{i}\right)^{p/2} \left(\frac{N^\alpha}{\sqrt{in}}\right)^{vp}. \end{aligned} \quad (5.22)$$

The last inequality follows from (5.8).

Let  $\varepsilon \in (0, 1)$ ,  $p := 4\varepsilon^{-1}$ , and  $b := (n/i)^{1/2} (N^\alpha/\sqrt{in})^v N^\varepsilon$  to obtain,

$$\begin{aligned} &\mathbf{P} \left( \bigcup_{i=1}^N \bigcup_{n=1}^N \left\{ \sup_{|x-y| \leq N^\alpha} |L_i(x; n) - L_i(y; n)| > \left(\frac{n}{i}\right)^{1/2} \left(\frac{N^\alpha}{\sqrt{in}}\right)^v N^\varepsilon \right\} \right) \\ &\leq N^2 \left( \frac{4c_9(v)}{\varepsilon} \right)^{4/\varepsilon} N^{-4}. \end{aligned} \quad (5.23)$$

Since this is summable in  $N$ , we can use the Borel–Cantelli lemma to see that with probability one, for all large  $N$  and all  $i, n \leq N$ ,

$$\sup_{|x-y| \leq N^\alpha} |L_i(x; n) - L_i(y; n)| \leq n^{1/2(1-v)} i^{-1/2(1+v)} N^{zv+\varepsilon}, \quad (5.24)$$

thus leading to our lemma.  $\square$

We are ready to present our

**Proof of Proposition 4.1.** According to [17, Theorem 2.2], the following holds with probability one:

$$\begin{aligned} &\int_0^1 \sqrt{u} \sup_{x \in \mathbf{R}} L_u(x; N) du = O(\sqrt{N \log \log N}) \\ &\sup_{1 \leq u \leq N} \sup_{x \in \mathbf{R}} \sqrt{u} L_u(x; N) = O(\sqrt{N \log \log N}). \end{aligned} \quad (5.25)$$

Let  $C(x; s, t)$  be the crossing local time of  $W$ , as in (2.4). For  $1 \leq m, n \leq N$ ,

$$\begin{aligned} C(x; m, n) &= \int_0^m \sqrt{u} L_u(x; n) du \\ &= O(\sqrt{N \log \log N}) + \int_1^m \sqrt{u} L_u(x; n) du \\ &= \sum_{i=1}^m \sqrt{i} L_i(x; n) + O(N^{1/2(3-v)} \log N). \end{aligned} \quad (5.26)$$

The last inequality following from Lemma 5.5 and (5.25), and  $O(\cdot \cdot)$  is uniform in  $1 \leq m, n \leq N$  and  $x \in \mathbf{R}$ . From this and Lemma 5.6, we can see that for any  $v \in [0, \frac{1}{2})$ , almost surely,

$$\sup_{|x-y| \leq N^\alpha} |C(x; m, n) - C(y; m, n)| = O(N^{1/2(3-v)} \log N + N^{zv+(3/2)-v+\varepsilon}), \quad (5.27)$$

uniformly in  $1 \leq m, n \leq N$ . Since  $\max(\frac{1}{2}(3-v), \alpha v + \frac{3}{2} - v + \varepsilon)$  can be as close to  $1 + \max(\frac{\alpha}{2}, \frac{1}{4})$  as possible, Proposition 4.1 follows.  $\square$

## 6. Proof of Proposition 4.2

This section is devoted to estimating the increments of  $x \mapsto Z(x; m, n) = \sum_{i=0}^m \zeta_i(x; n)$ . By the definition of  $\zeta_i(x; n)$  in (2.2),

$$\zeta_i(x; n) = \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > x, S_{i,j+1} < x\}} + \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} < x, S_{i,j+1} > x\}}. \quad (6.1)$$

The two sums on the right-hand side represent the numbers of down- and up-crossings, respectively. Thus,

$$\left| \zeta_i(x; n) - 2 \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > x, S_{i,j+1} < x\}} \right| \leq 1. \quad (6.2)$$

It remains to study the increments of  $x \mapsto \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > x, S_{i,j+1} < x\}}$ . Our first step is to estimate  $S_{ij}$  by a Wiener process. An obvious candidate would be the Brownian sheet in (3.6) that was used in Section 3 to prove a strong approximation of  $S_{ij}$ . Unfortunately, the error term in this approximation scheme is too large for our needs. So, we proceed very carefully in replacing  $S_{ij}$  by another Brownian motion.

Fix  $i \leq N$ . Since  $S_{ij}$  is the sum of  $(ij)$  i.i.d. symmetric Bernoulli random variables, by the KMT theorem (3.2) there exists a standard Wiener process  $\{W(t); t \geq 0\}$  such that for any  $z > 0$  and some absolute constants  $c_1, c_2$  and  $c_3$ ,

$$\mathbf{P}\{|S_{ij} - W(ij)| > c_1 \log(ij) + z\} \leq c_2 e^{-c_3 z}. \quad (6.3)$$

Strictly speaking, we really should write  $W_i$  instead of  $W$  (our Wiener process  $W$  depends on  $i$ ). The same remark applies to the forthcoming event  $A$  and Wiener process  $B$ .

Fix  $\varepsilon \in (0, \frac{1}{2})$  and  $\delta \in (0, \frac{\varepsilon}{2})$ , and consider the event

$$E := \bigcap_{j=1}^N \{|S_{ij} - W(ij)| \leq N^\delta\} \quad \forall i \leq N. \quad (6.4)$$

By (6.3), for large  $N$ , say  $N \geq N_0$ ,

$$\mathbf{P}(E^c) \leq \sum_{j=1}^N c_{17} N^{-5} = c_{17} N^{-4}. \quad (6.5)$$

On the event  $E$ , we have for  $i \leq N$ ,  $n \leq N$  and  $x \in \mathbf{R}$ ,

$$\begin{aligned} \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > x, S_{i,j+1} < x\}} &\leq \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij) > x - N^\delta, W(i(j+1)) < x + N^\delta\}} \\ &\leq \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij) > x, W(i(j+1)) < x\}} + 2 \sup_{a \in \mathbf{R}} \sum_{j=0}^n \mathbf{1}_{\{a \leq W(ij) \leq a + N^\delta\}}. \end{aligned} \quad (6.6)$$



Similarly, on  $E$ ,

$$\sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > x, S_{i,j+1} < x\}} \geq \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij) > x, W(i(j+1)) < x\}} - 2 \sup_{a \in \mathbf{R}} \sum_{j=0}^n \mathbf{1}_{\{a \leq W(ij) \leq a + N^\delta\}}. \quad (6.7)$$

If we write

$$A_{i,n} = \sup_{|x-y| \leq N^\alpha} \left| \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > x, S_{i,j+1} < x\}} - \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{ij} > y, S_{i,j+1} < y\}} \right|, \quad (6.8)$$

then

$$\begin{aligned} A_{i,n} \mathbf{1}_E &\leq \sup_{|x-y| \leq N^\alpha} \left| \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij) > x, W(i(j+1)) < x\}} - \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij) > y, W(i(j+1)) < y\}} \right| \\ &\quad + 4 \sup_{a \in \mathbf{R}} \sum_{j=0}^n \mathbf{1}_{\{a \leq W(ij) \leq a + N^\delta\}}. \end{aligned} \quad (6.9)$$

Let us consider the process  $x \mapsto \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij) > x, W(i(j+1)) < x\}}$ : For each  $i$ , it is distributed as  $x \mapsto \sum_{j=0}^{n-1} \mathbf{1}_{\{W(j) > x/\sqrt{i}, W(j+1) < x/\sqrt{i}\}}$ . Therefore, by [4, Eq. (1.13)], there exists a coupling of  $W$  and a standard Wiener process  $B$  such that for any  $\lambda > 1$ , there exist constants  $c_{18}$  and  $c_{19}$  such that for all  $n \geq 1$ ,

$$\begin{aligned} P \left\{ \sup_{x \in \mathbf{R}} \left| \sum_{j=0}^{n-1} \mathbf{1}_{\{W(j) > x, W(j+1) < x\}} - c_{20} \sqrt{n} L \left( \frac{x}{\sqrt{n}}, 1 \right) \right| \geq c_{18} n^{1/4} \log n \right\} \\ \leq c_{19} n^{-\lambda}, \end{aligned} \quad (6.10)$$

where  $L$  is the local time of  $B$ , and  $c_{20} := E(\lfloor B^+(1) \rfloor)$ . We also mention that in [4, Eq. (1.13)], the preceding inequality was stated only for some  $\lambda > 1$ , which sufficed for the intended applications there. However, it is clear from its proof [4, pp. 272–273] that by altering  $c_{19} = c_{19}(\lambda)$  suitably,  $\lambda$  can be made to be arbitrarily large.

A straightforward consequence of (6.9) and (6.10), with  $\lambda = \frac{4}{\delta}$  in (6.10), is as follows: For any  $b > 0$  and  $k > 0$ ,

$$\begin{aligned} P\{A_{i,n} \mathbf{1}_E > b + k + 2c_{18} n^{1/4} \log n\} \\ \leq P \left\{ \sup_{|x-y| \leq N^\alpha} \left| c_{20} \sqrt{n} L \left( \frac{x}{\sqrt{in}}, 1 \right) - c_{20} \sqrt{n} L \left( \frac{y}{\sqrt{in}}, 1 \right) \right| > b \right\} \\ + P \left\{ 4 \sup_{a \in \mathbf{R}} \sum_{j=0}^n \mathbf{1}_{\{a \leq W(ij) \leq a + N^\delta\}} > k \right\} + c_{19} n^{-4/\delta} \\ = P_1 + P_2 + c_{19} n^{-4/\delta}. \end{aligned} \quad (6.11)$$

Plugging this into (6.5) yields

$$P\{A_{i,n} > b + k + 2c_{18} n^{1/4} \log n\} \leq P_1 + P_2 + c_{19} n^{-4/\delta} + c_{17} N^{-4}. \quad (6.12)$$

To bound  $P_1$ , we use (5.8) to see that for any  $v \in [0, \frac{1}{2}]$  and  $p \geq 1$ ,

$$\begin{aligned} P_1 &= \mathbf{P} \left\{ \sup_{|x-y| \leq N^\alpha / \sqrt{in}} |L(x, 1) - L(y, 1)| > \frac{b}{2c_{20}\sqrt{n}} \right\} \\ &\leq \mathbf{P} \left\{ \sup_{x \neq y} \frac{|L(x, 1) - L(y, 1)|}{|x - y|^v} > \frac{b}{2c_{20}\sqrt{n}(N^\alpha / \sqrt{in})^v} \right\} \\ &\leq (pc_9(v))^p \left( \frac{2c_{20}\sqrt{n}(N^\alpha / \sqrt{in})^v}{b} \right)^p. \end{aligned} \quad (6.13)$$

Choosing  $b = \sqrt{n}(N^\alpha / \sqrt{in})^v N^\delta$  and  $p = \frac{4}{\delta}$  gives

$$P_1 \leq c_{21} N^{-4}, \quad (6.14)$$

where  $c_{21} = (8c_{20} c_9(v)/\delta)^{4/\delta}$  depends only on  $(v, \delta)$ .

We now estimate  $P_2$ : The standard Gaussian density is bounded above by  $(2\pi)^{-1/2} \leq \frac{1}{2}$ . Therefore, for any  $a \in \mathbf{R}$ , we have  $\mathbf{P}\{a \leq W(ij) \leq a + 2N^\delta\} \leq N^\delta (ij)^{-1/2}$ . We can apply this together with the Markov property to deduce that for any integer  $p \geq 1$ ,

$$\left\| \sum_{j=1}^n \mathbf{1}_{\{a \leq W(ij) \leq a + 2N^\delta\}} \right\|_p^p \leq p! \left( \sum_{j=1}^n \frac{N^\delta}{\sqrt{ij}} \right)^p \leq c_{22} \left( \frac{N^\delta \sqrt{n}}{\sqrt{i}} \right)^p, \quad (6.15)$$

for some constant  $c_{22} = c_{22}(p)$  that depends only on  $p$ . By Markov's inequality, for any  $k > 0$ ,

$$\sup_{a \in \mathbf{R}} \mathbf{P} \left\{ \sum_{j=1}^n \mathbf{1}_{\{a \leq W(ij) \leq a + 2N^\delta\}} > k \right\} \leq \frac{c_{22}}{k^p} \left( \frac{N^\delta \sqrt{n}}{\sqrt{i}} \right)^p. \quad (6.16)$$

Now let  $a_\ell = 2N^\delta \ell$ , for  $\ell = 0, \pm 1, \pm 2, \dots, \pm \ell_{\max}$ , where  $\ell_{\max} = \lfloor N^\delta \sqrt{in} \rfloor$ . By the usual estimate for Gaussian tails,

$$\mathbf{P} \left\{ \max_{1 \leq j \leq n} |W(ij)| > a_{\ell_{\max}} \right\} \leq 2 \exp \left( - \frac{(2N^\delta \lfloor N^\delta \sqrt{in} \rfloor)^2}{2in} \right) \leq 2 \exp(-N^{4\delta}). \quad (6.17)$$

On the other hand, if  $\max_{1 \leq j \leq n} |W(ij)| > a_{\ell_{\max}}$ , and if  $\sum_{j=1}^n \mathbf{1}_{\{a \leq W(ij) \leq a + N^\delta\}} > k$  for some  $a \in \mathbf{R}$ , then there exists  $\ell$ , with  $|\ell| \leq \ell_{\max}$  such that  $\sum_{j=1}^n \mathbf{1}_{\{a_\ell \leq W(ij) \leq a_\ell + 2N^\delta\}} > k$ . By (6.16), this yields,

$$\begin{aligned} &\mathbf{P} \left\{ \sup_{a \in \mathbf{R}} \sum_{j=1}^n \mathbf{1}_{\{a \leq W(ij) \leq a + N^\delta\}} > k \right\} \\ &\leq (2\ell_{\max} + 1) \frac{c_{22}}{k^p} \left( \frac{N^\delta \sqrt{n}}{\sqrt{i}} \right)^p + 2 \exp(-N^{4\delta}). \end{aligned} \quad (6.18)$$

We replace  $k$  by  $\frac{1}{4}k$  to deduce that for all  $i, n \leq N$ ,

$$\begin{aligned} P_2 &\leq c_{23} \frac{N^\delta \sqrt{in}}{k^p} \left( \frac{N^\delta \sqrt{n}}{\sqrt{i}} \right)^p + 2 \exp(-N^{4\delta}) \\ &\leq c_{23} \frac{N^\delta \sqrt{in}}{k^p} \left( \frac{N^\delta \sqrt{n}}{\sqrt{i}} \right)^p + 2 \exp(-N^{4\delta}). \end{aligned} \quad (6.19)$$

Taking  $k = N^{2\delta}(n/i)^{1/2}$  and  $p = \frac{1}{\delta}(5 + \delta)$  yields

$$P_2 \leq c_{23} N^{-4} + 2 \exp(-N^{4\delta}). \quad (6.20)$$

Plugging this and (6.14) into (6.12) implies that for  $i, n \leq N$ ,

$$\begin{aligned} P \left\{ \Delta_{i,n} > \sqrt{n} \left( \frac{N^\alpha}{\sqrt{in}} \right)^v N^\delta + N^{2\delta} \sqrt{\frac{n}{i}} + 2c_{18} n^{1/4} \log n \right\} \\ \leq c_{24} N^{-4} + 2 \exp(-N^{4\delta}) + c_{19} n^{-4/\delta}, \end{aligned} \quad (6.21)$$

where  $c_{24} := c_{21} + c_{23} + c_{17}$ . As a consequence,

$$\begin{aligned} P \left( \bigcup_{i=1}^N \bigcup_{n=\lfloor N^\delta \rfloor}^N \left\{ \Delta_{i,n} > \sqrt{n} \left( \frac{N^\alpha}{\sqrt{in}} \right)^v N^\delta + N^{2\delta} \sqrt{\frac{n}{i}} + 2c_{18} n^{1/4} \log n \right\} \right) \\ \leq c_{24} N^{-2} + 2N^2 \exp(-N^{4\delta}) + c_{19} N^{-2}, \end{aligned} \quad (6.22)$$

which is summable for  $N$ . By the Borel–Cantelli lemma, almost surely for all large  $N$ ,  $i \leq N$ , and every  $N^\delta \leq n \leq N$ ,

$$\Delta_{i,n} \leq \sqrt{n} \left( \frac{N^\alpha}{\sqrt{in}} \right)^v N^\delta + N^{2\delta} \sqrt{\frac{n}{i}} + 2c_{18} n^{1/4} \log n. \quad (6.23)$$

Thus, whenever  $m \leq N$  and  $N^\delta \leq n \leq N$ ,

$$\begin{aligned} \sum_{i=1}^m \Delta_{i,n} &\leq \sqrt{n} N^{1-(v/2)} \left( \frac{N^\alpha}{\sqrt{n}} \right)^v N^\delta + N^{2\delta+(1/2)} \sqrt{n} + N^{(5/4)+\delta} \\ &\leq N^{(3/2)-v+\alpha v+\delta} + 2N^{(5/4)+\delta}, \quad \text{eventually (in } N). \end{aligned} \quad (6.24)$$

This last inequality uses the facts that  $n \leq N$  and that  $\delta < \frac{1}{4}$ .

If  $n < N^\delta$ , we can use the trivial inequality  $\Delta_{i,n} \leq 2n$  to see that  $\sum_{i=1}^m \Delta_{i,n} \leq 2N^{1+\delta}$ . Therefore, whenever  $v \in [0, \frac{1}{2})$ ,  $\varepsilon \in (0, \frac{1}{4})$  and  $\delta \in (0, \frac{\varepsilon}{2})$ , with probability one, for all large  $N$  and  $m, n \leq N$ ,

$$\sum_{i=1}^m \Delta_{i,n} \leq N^{(3/2)-v+\alpha v+\delta} + 3N^{(5/4)+\delta}. \quad (6.25)$$

Since  $v \in [0, \frac{1}{2})$  is arbitrary, it can be chosen such that  $\frac{3}{2} - v + \alpha v + \delta < 1 + \frac{\alpha}{2} + 2\delta$ . In view of the definition of  $\Delta_{i,n}$  in (6.8), and relation (6.2), we have proved that, almost

surely, for any  $\varepsilon \in (0, \frac{1}{2})$ , when  $N \rightarrow \infty$ ,

$$\begin{aligned} & \max_{0 \leq m \leq N} \max_{1 \leq n \leq N} \sup_{|x-y| \leq N^\alpha} \left| \sum_{i=0}^m \zeta_i(x; n) - \sum_{i=0}^m \zeta_i(y; n) \right| \\ &= o(N^{1+(\alpha/2)+\varepsilon}) + N^{(5/4)+\varepsilon}. \end{aligned} \quad (6.26)$$

This completes our proof of Proposition 4.2.  $\square$

## 7. Proof of Theorem 1.1

In view of (4.10), we need to check only the lower bound; namely that for any  $\varepsilon > 0$ ,  $Z(N) \geq N^{(3/2)-\varepsilon}$  almost surely for all large  $N$ . By means of (2.5), we have to prove that for large  $N$ ,

$$C(0; N, N) \geq N^{(3/2)-\varepsilon}, \quad \text{a.s.} \quad (7.1)$$

We start by choosing  $a \in (\frac{1}{2}, 1)$  and  $b > 1$  such that  $2a > b$ . We also recall that  $C(0; N, N) = \int_0^N \sqrt{u} L_u(0; N) du$ . Thus,

$$C(0; N, N) \geq \int_{aN}^N \sqrt{aN} L_u(0; N) du \geq \sqrt{a}(1-a)N^{3/2} \inf_{aN \leq u \leq N} L_u(0; N). \quad (7.2)$$

We now use the following recent estimate of the authors [15, Theorem 3.3]: For any  $v \in (0, \frac{1}{2})$ , there exists a small constant  $h_0 = h_0(v, a)$  such that for all  $N > 0$  and  $h \in (0, h_0)$ ,

$$\mathbf{P} \left\{ \inf_{\frac{1}{2}N \leq u \leq N} L_u \left( 0; \frac{N}{2a} \right) \leq h \right\} \leq \exp \left( - \frac{v |\log h|}{\log |\log h|} \right). \quad (7.3)$$

In [15] we obtained the above for  $N = 1$ . This formulation is a consequence of the case  $N = 1$  and scaling. Taking  $N_k = \lfloor b^k \rfloor$  and  $h = N_k^{-\varepsilon/2}$  yields

$$\sum_{k=1}^{\infty} \mathbf{P} \left\{ \inf_{\frac{1}{2}N_k \leq u \leq N_k} L_u \left( 0; \frac{N_k}{2a} \right) \leq N_k^\varepsilon \right\} < +\infty. \quad (7.4)$$

By the Borel–Cantelli lemma, almost surely for all large  $k$ ,

$$\inf_{\frac{1}{2}N_k \leq u \leq N_k} L_u \left( 0; \frac{N_k}{2a} \right) > N_k^{-\varepsilon/2}. \quad (7.5)$$

Consequently, whenever  $N_{k-1} \leq N \leq N_k$ ,

$$\begin{aligned} \inf_{aN \leq u \leq N} L_u(0; N) &\geq \inf_{aN_{k-1} \leq u \leq N_k} L_u(0; N_{k-1}) \geq \inf_{\frac{1}{2}N_k \leq u \leq N_k} L_u \left( 0; \frac{N_k}{2a} \right) \\ &> N_k^{-\varepsilon/2} \geq (2aN)^{-\varepsilon/2}. \end{aligned} \quad (7.6)$$

We have used the fact that  $N_{k-1} \geq (2a)^{-1} N_k$  for all sufficiently large  $k$ ; this, in turn, follows from the inequality  $2a > b$ . In summary, for all large  $N$ ,

$$C(0; N, N) \geq \sqrt{a}(1-a)N^{3/2}(2aN)^{-\varepsilon/2} \geq N^{3/2-\varepsilon}, \quad (7.7)$$

yielding (7.1), whence Theorem 1.1.  $\square$

## Appendix A. The Matlab code

The Matlab code for Fig. 1(a) follows. It presupposes the existence of a seed for the random number generator, and that the seed is stored in a Matlab file called “seed.dat.”

```

1.  % run a 1-dimensional simple walk for n time-steps
2.  load seed    % The same random walk is always used
3.  rand('state', seed)
4.  figure;
5.  x = rand(n);
6.  X = 2*round(x)-1;          % Generate Rademacher variables
7.  S=cumsum(X);              % Sum the columns separately
8.  W=S;
9.  W(:,1) = S(:,1);
10. for i=2:n
        W(:,i) = W(:,i-1) + S(:,i);          % W is the walk
    end
11. for k=1:n
        for j=1:n
            if W(k,j) == 0
                plot(k,j)
            end
        end
    end
12. print -deps file.ps

```

In order to simulate the vertical crossings (for the same walk), the third line of 11 above (i.e., “if W(k,j)==0”) needs to be replaced by “if W(k,j)\*W(k,j+1) < 0.”

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